Topos theory and 'neo-realist' quantum theory

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Topos theory, a branch of category theory, has been proposed as mathematical basis for the formulation of physical theories. In this article, we give a brief introduction to this approach, emphasising the logical aspects. Each topos serves as a 'mathematical universe' with an internal logic, which is used to assign truth-values to all propositions about a physical system. We show in detail how this works for (algebraic) quantum theory.

"The problem is all inside your head", she said to me
the answer is easy if you take it logically

Paul Simon (from *'50 Ways To Leave Your Lover'*)

1 Introduction

The use of topos theory in the foundations of physics and, in particular, the foundations of quantum theory was suggested by Chris Isham more than 10 years ago in [Ish97]. Subsequently, these ideas were developed in an application to the Kochen-Specker theorem (with Jeremy Butterfield, [IB98, IB99, IBH00, IB02], for conceptual considerations see [IB00]). In these papers, the use of a multi-valued, contextual logic for quantum theory was proposed. This logic is given by the internal logic of a certain topos of presheaves over a category of contexts. Here, contexts typically are abelian parts of a larger, non-abelian structure. There are several possible choices for the context category. We will concentrate on algebraic quantum theory and use the category $\mathcal{V}(\mathcal{R})$ of unital abelian von Neumann

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subalgebras of the non-abelian von Neumann algebra of observables $\mathcal{R}$ of the quantum system, as first suggested in [IBH00].

The use of presheaves over such a category of contexts is motivated by the very natural construction of the spectral presheaf $\underline{\Sigma}$, which collects all the Gel'fand spectra of the abelian subalgebras $V \in \mathcal{V}(\mathcal{R})$ into one larger structure. The Gel'fand spectra can be seen as 'local state spaces', and the spectral presheaf serves as a state space analogue for quantum theory. Interestingly, as Isham and Butterfield showed, this presheaf is not like a space: it has no points (in a category-theoretical sense), and this fact is exactly equivalent to the Kochen-Specker theorem.

The topos approach was developed considerably in the series of papers [DI07a, DI07b, DI07c, DI07d] by Chris Isham and the author. In these papers, it was shown how topos theory can serve as a new mathematical framework for the formulation of physical theories. The basic idea of the topos programme is that by representing the relevant physical structures (states, physical quantities and propositions about physical quantities) in suitable topoi, one can achieve a remarkable structural similarity between classical and quantum physics. Moreover, the topos programme is general enough to allow for major generalisations. Theories beyond classical and quantum theory are conceivable. Arguably, this generality will be needed in a future theory of quantum gravity, which is expected to go well beyond our conventional theories.

In this paper, we will concentrate on algebraic quantum theory. We briefly motivate the mathematical constructions and give the main definitions.[1] Throughout, we concentrate on the logical aspects of the theory. We will show in detail how, given a state, truth-values are assigned to all propositions about a quantum system. This is independent of any measurement or observer. For that reason, we say that the topos approach gives a 'neo-realist' formulation of quantum theory.

1.1 What is a topos?

It is impossible to give even the briefest introduction to topos theory here. At the danger of being highly imprecise, we restrict ourselves to mentioning some aspects of this well-developed mathematical theory and give a number of pointers to the literature. The aim merely is to give a very rough idea of the structure and internal logic of a topos. In the next subsection, we argue that this mathematical structure may be useful in physics.

There are a number of excellent textbooks on topos theory, and the reader should consult at least one of them. We found the following books useful: [LR03, Gol84, MM92, Joh02a, Joh02b, Bell88, LS86].

[1] We suppose that the reader is familiar with the definitions of a category, functor and natural transformation.

Topoi as mathematical universes. Every (elementary) topos $\mathcal{E}$ can be seen as a *mathematical universe*. As a category, a topos $\mathcal{E}$ possesses a number of structures that generalise constructions that are possible in the category **Set** of sets and functions.[2] Namely, in **Set**, we can construct new sets from given ones in several ways: let S, T be two sets, then we can form the cartesian product $S \times T$, the disjoint union $S \amalg T$ and the exponential S^T, the set of all functions from T to S. These constructions turn out to be fundamental and can all be phrased in an abstract, categorical manner, where they are called finite limits, colimits and exponentials, respectively. By definition, a topos $\mathcal{E}$ has all of these. One consequence of the existence of finite limits is that each topos has a *terminal object*, denoted by 1. This is characterised by the property that for any object A in the topos $\mathcal{E}$, there exists exactly one arrow from A to 1. In **Set**, a one-element set $1 = \{*\}$ is terminal.[3]

Of course, **Set** is a topos, too, and it is precisely the topos which usually plays the rôle of our mathematical universe, since we construct our mathematical objects starting from sets and functions between them. As a slogan, we have: a topos $\mathcal{E}$ is a category similar to **Set**. A very nice and gentle introduction to these aspects of topos theory is the book [LR03]. Other good sources are [Gol84, McL71].

In order to 'do mathematics', one must also have a logic, including a deductive system. Each topos comes equipped with an *internal logic*, which is of *intuitionistic* type. We very briefly sketch the main characteristics of intuitionistic logic and the mathematical structures in a topos that realise this logic.

Intuitionistic logic. Intuitionistic logic is similar to Boolean logic, the main difference being that the *law of excluded middle* need not hold. In intuitionistic logic, there is *no* axiom

$$\vdash a \vee \neg a \tag{$*$}$$

like in Boolean logic. Here, $\neg a$ is the negation of the formula (or proposition) a. The algebraic structures representing intuitionistic logic are *Heyting algebras*. A Heyting algebra is a pseudocomplemented, distributive lattice[4] with zero element 0 and unit element 1, representing 'totally false' resp. 'totally true'. The pseudocomplement is denoted by $\neg$, and one has, for all elements α of a Heyting algebra H,

$$\alpha \vee \neg\alpha \leq 1,$$

in contrast to $\alpha \vee \neg\alpha = 1$ in a Boolean algebra. This means that the disjunction ("Or") of a proposition α and its negation need not be (totally) true in a Heyting algebra. Equivalently,

[2] More precisely, *small* sets and functions between them. Small means that we do not have proper classes. One must take care in these foundational issues to avoid problems like Russell's paradox.

[3] Like many categorical constructions, the terminal object is fixed only up to isomorphism: any two one-element sets are isomorphic, and any of them can serve as a terminal object. Nonetheless, one speaks of *the* terminal object.

[4] Lattice is meant in the algebraic sense: a partially ordered set L such that any two elements $a, b \in L$ have a minimum (greatest lower bound) $a \wedge b$ and a maximum (least upper bound) $a \vee b$ in L. A lattice L is distributive if and only if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ as well as $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ hold for all $a, b, c \in L$.

one has

$$\neg\neg\alpha \geq \alpha,$$

in contrast to $\neg\neg\alpha = \alpha$ in Boolean algebras.

Obviously, Boolean logic is a special case of intuitionistic logic. It is known from Stone's theorem [Sto36] that each Boolean algebra is isomorphic to an algebra of (clopen, i.e., closed and open) subsets of a suitable (topological) space.

Let X be a set, and let $P(X)$ be the power set of X, that is, the set of subsets of X. Given a subset $S \in P(X)$, one can ask for each point $x \in X$ whether it lies in S or not. This can be expressed by the *characteristic function* $\chi_S : X \to \{0, 1\}$, which is defined as

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

for all $x \in X$. The two-element set $\{0, 1\}$ plays the rôle of a set of *truth-values* for propositions (of the form "$x \in S$"). Clearly, 1 corresponds to 'true', 0 corresponds to 'false', and there are no other possibilities. This is an argument about sets, so it takes place in and uses the logic of the topos **Set** of sets and functions. **Set** is a *Boolean topos*, in which the familiar two-valued logic and the axiom $(*)$ hold. (This does not contradict the fact that the internal logic of topoi is intuitionistic, since Boolean logic is a special case of intuitionistic logic.)

In an arbitrary topos, there is a special object Ω, called the *subobject classifier*, that takes the rôle of the set $\{0, 1\} \simeq \{\text{false,true}\}$ of truth-values. Let B be an object in the topos, and let A be a subobject of B. This means that there is a monic $A \to B$,[5] generalising the inclusion of a subset S into a larger set X. Like in **Set**, we can also characterise A as a subobject of B by an arrow from B to the subobject classifier Ω. (In **Set**, this arrow is the characteristic function $\chi_S : X \to \{0, 1\}$.) Intuitively, this characteristic arrow from B to Ω tells us how A 'lies in' B. The textbook definition is:

Definition 1 *In a category $\mathcal{C}$ with finite limits, a **subobject classifier** is an object Ω, together with a monic* $\text{true} : 1 \to \Omega$*, such that to every monic $m : A \to B$ in $\mathcal{C}$ there is a unique arrow χ which, with the given monic, forms a pullback square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "\text{true}"] \\
B \arrow[r, "\chi"'] & \Omega
\end{tikzcd}$$

[5] A *monic* is the categorical version of an injective function. In the topos **Set**, monics exactly are injective functions.

In **Set**, the arrow true : $1 \to \{0,1\}$ is given by true$(*) = 1$. In general, the subobject classifier Ω need not be a set, since it is an object in the topos $\mathcal{E}$, and the objects of $\mathcal{E}$ need not be sets. Nonetheless, there is an abstract notion of *elements* (or *points*) in category theory that we can use. The elements of Ω are the truth-values available in the internal logic of our topos $\mathcal{E}$, just like 'false' and 'true', the elements of {false,true}, are the truth-values available in the topos **Set**.

To understand the abstract notion of elements, let us consider sets for a moment. Let $1 = \{*\}$ be a one-element set, the terminal object in **Set**. Let S be a set and consider an arrow e from 1 to S. Clearly, $e(*) \in S$ is one element of S. The set of all functions from 1 to S corresponds exactly to the elements of S. This idea can be generalised to other categories: if there is a terminal object 1, then we consider arrows from 1 to an object A in the category as *elements of* A. For example, in the definition of the subobject classifier the arrow true : $1 \to \Omega$ is an element of Ω. It may happen that an object A has no elements, i.e., there are no arrows $1 \to A$. It is common to consider arrows from subobjects U of A to A as *generalised elements*, but we will not need this except briefly in subsection 5.1.

As mentioned, the elements of the subobject classifier, understood as the arrows $1 \to \Omega$, are the truth-values. Moreover, the set of these arrows forms a Heyting algebra (see, for example, section 8.3 in [Gol84]). This is how (the algebraic representation of) intuitionistic logic manifests itself in a topos. Another, closely related fact is that the subobjects of any object A in a topos form a Heyting algebra.

1.2 Topos theory and physics

A large part of the work on topos theory in physics consists in showing how states, physical quantities and propositions about physical quantities can be represented within a suitable topos attached to the system [DI07a, DI07b, DI07c, DI07d]. The choice of topos will depend on the theory type (classical, quantum or, in future developments, even something completely new). Let us consider classical physics for the moment to motivate this.

Realism in classical physics. In classical physics, one has a space of states $\mathcal{S}$, and physical quantities A are represented by real-valued functions $f_A : \mathcal{S} \to \mathbb{R}$.[6] A proposition about a physical quantity A is of the form "$A \in \Delta$", which means "the physical quantity A has a value in the (Borel) set Δ". This proposition is represented by the inverse image $f_A^{-1}(\Delta) \subseteq \mathcal{S}$. In general, propositions about the physical system correspond to Borel subsets of the state space $\mathcal{S}$. If we have two propositions "$A \in \Delta_1$", "$B \in \Delta_2$" and the corresponding subsets $f_A^{-1}(\Delta_1)$, $f_B^{-1}(\Delta_2)$, then the intersection $f_A^{-1}(\Delta_1) \cap f_B^{-1}(\Delta_2)$ corresponds to the proposition "$A \in \Delta_1$ and $B \in \Delta_2$", the union $f_A^{-1}(\Delta_1) \cup f_B^{-1}(\Delta_2)$ corresponds to "$A \in \Delta_1$ or $B \in \Delta_2$", and the complement $\mathcal{S} \backslash f_A^{-1}(\Delta_1)$ corresponds to the negation "$A \notin \Delta_1$". Moreover, given a state s, i.e., an element of the state space $\mathcal{S}$, each proposition is either true or false: if s lies in the subset of $\mathcal{S}$ representing the proposition,

[6] We assume that f_A is at least measurable.

then the proposition is true, otherwise it is false. Every physical quantity A has a value in the state s, namely $f_A(s) \in \mathbb{R}$. Thus classical physics is a *realist* theory in which propositions have truth-values independent of measurements, observers etc. The logic is Boolean, since classical physics is based on constructions with sets and functions, i.e., it takes place in the topos **Set**. We take this as a rule: if we want to describe a physical system S as a classical system, then the topos **Set** is used. This means no departure from what is ordinarily done, but it emphasises certain structural and logical aspects of the theory.

Instrumentalism in quantum theory. In quantum theory, the mathematical description is very different. Physical quantities A are represented by self-adjoint operators $\widehat{A}$ on a Hilbert space $\mathcal{H}$. While $\mathcal{H}$ can be called a space of states, the states $\psi \in \mathcal{H}$ play a very different rôle from those in classical theory. In particular, a state ψ does not assign values to all physical quantities, only to those for which ψ happens to be an eigenstate. The spectral theorem shows that propositions "$A \in \Delta$" are represented by projection operators $\widehat{E}[A \in \Delta]$ on Hilbert space. Unless ψ is an eigenstate of A, such a proposition is neither true nor false (except for the trivial cases $\widehat{E}[A \in \Delta] = \widehat{0}$, which represents trivially false propositions, and $\widehat{E}[A \in \Delta] = \widehat{1}$, which represents trivially true propositions). The mathematical formalism of quantum theory is interpreted in an *instrumentalist* manner: given a state ψ, the proposition "$A \in \Delta$" is assigned a probability of being true, given by the expectation value $p(A \in \Delta; \psi) := \langle\psi|\, \widehat{E}[A \in \Delta]\, |\psi\rangle$. This means that upon measurement of the physical quantity A, one will find the measurement result to lie in Δ with probability $p(A \in \Delta; \psi)$. This interpretation depends on measurements and an external observer. Moreover, the measurement devices (and the observer) are described in terms of classical physics, not quantum physics.

The motivation from quantum gravity. An instrumentalist interpretation cannot describe closed quantum systems, at least there is nothing much to be said about them from this perspective. A theory of quantum cosmology or quantum gravity will presumably be a quantum theory of the whole universe. Since there is no external observer who could perform measurements in such a theory, instrumentalism becomes meaningless. One of the main motivations for the topos programme is to overcome or circumvent the usual instrumentalism of quantum theory and to replace it with a more realist account of quantum systems. The idea is to use the internal logic of a topos to assign truth-values to propositions about the system.

In order to achieve this, we will sketch a new mathematical formulation of quantum theory that is structurally similar to classical physics. The details can be found in [DI07a, DI07b, DI07c, DI07d] and references therein.

Plan of the paper. The starting point is the definition of a formal language $\mathcal{L}(S)$ attached to a physical system S. This is done in section 2 and emphasises the common structure of classical and quantum physics. In section 3, we introduce the topos associated to a system S in the case of quantum theory, and in section 4 we briefly discuss the representation of $\mathcal{L}(S)$ in this topos. The representation of states and the assignment

of truth-values to propositions is treated in section 5, which is the longest and most detailed section. Section 6 concludes with some remarks on related work and on possible generalisations.

2 A formal language for physics

There is a well-developed branch of topos theory that puts emphasis on the logical aspects. As already mentioned, a topos can be seen as the embodiment of (higher-order) intuitionistic logic. This point of view is expounded in detail in Bell's book [Bell88], which is our standard reference on these matters. Other excellent sources are [LS86] and part D of [Joh02b]. The basic concept consists in defining a *formal language* and then finding a *representation* of it in a suitable topos. As usual in mathematical logic, the formal language encodes the syntactic aspects of the theory and the representation provides the semantics. Topoi are a natural 'home' for the representation of formal languages encoding intuitionistic logic, more precisely, *intuitionistic, higher-order, typed predicate logic with equality.* Typed means that there are several primitive species or kinds of objects (instead of just sets as primitives), from which sets are extracted as a subspecies; predicate logic means that one has quantifiers, namely an existence quantifier $\exists$ ("it exists") and a universal quantifier $\forall$ ("for all"). Higher-order refers to the fact that quantification can take place not only over variable individuals, but also over subsets and functions of individuals as well as iterates of these constructions. Bell presents a particularly elegant way to specify a formal language with these properties. He calls this type of language a *local language*, see chapter 3 of [Bell88].

Let S denote a physical system to which we attach a higher-order, typed language $\mathcal{L}(S)$. We can only sketch the most important aspects here, details can be found in section 4 of [DI07a]. The language $\mathcal{L}(S)$ does not depend on the theory type (classical, quantum, ...), while its representation of course does. The language contains at least the following type symbols: $1, \Omega, \Sigma$ and $\mathcal{R}$. The symbol Σ serves as a precursor of the *state object* (see below), the symbol $\mathcal{R}$ is a precursor of the *quantity-value object*, which is where physical quantities take their values. Moreover, we require the existence of *function symbols* of the form $A : \Sigma \to \mathcal{R}$. These are the linguistic precursors of physical quantities. For each type, there exist *variables* of that type. There are a number of rules how to form terms and formulae (terms of type Ω) from variables of the various types, including the definition of logical connectives $\wedge$ ("And"), $\vee$ ("Or") and $\neg$ ("Not"). Moreover, there are axioms giving *rules of inference* that define how to get new formulae from sets of given formulae. As an example, we mention the *cut rule*: if Γ is a set of formulae and α and β are formulae, then we have

$$\frac{\Gamma : \alpha \quad \alpha, \Gamma : \beta}{\Gamma : \beta}$$

(here, any free variable in α must be free in Γ or β). In a representation, where the formulae aquire an interpretation and a 'meaning', this expresses that if Γ implies α, and α and Γ together imply β, then Γ also implies β. The axioms and rules of inference are chosen in a way such that the logical operations satisfy the laws of intuitionistic logic.

The formal language $\mathcal{L}(S)$ captures a number of abstract properties of the physical system S. For example, if S is the harmonic oscillator, then we expect to be able to speak about the physical quantity energy in all theory types, classical or quantum (or other). Thus, among the function symbols $A : \Sigma \to \mathcal{R}$, there will be one symbol $E : \Sigma \to \mathcal{R}$ which, in a representation, will become the mathematical entity describing energy. (Which mathematical object that will be depends on the theory type and thus on the representation.)

The representation of the language $\mathcal{L}(S)$ takes place in a suitable, physically motivated topos $\mathcal{E}$. The type symbol 1 is represented by the terminal object 1 in $\mathcal{E}$, the type symbol Ω is represented by the subobject classifier Ω. The choice of an appropriate object Σ in the topos that represents the symbol Σ depends on physical insight. The representing object Σ is called the *state object*, and it plays the rôle of a generalised state space. What actually is generalised is the space, not the states: Σ is an object in a topos $\mathcal{E}$, which need not be a topos of sets, so Σ need not be a set or space-like. However, as an object in a topos, Σ does have subobjects. These subobjects will be interpreted as (the representatives of) propositions about the physical quantities, just like in classical physics, where propositions correspond to subsets of state space. The propositions are of the form "$A \in \Delta$", where Δ now is a subobject of the object $\mathcal{R}$ that represents the symbol $\mathcal{R}$. The object $\mathcal{R}$ is called the *quantity-value object*, and this is where physical quantities take their values. Somewhat surprisingly, even for ordinary quantum theory this is *not* the real number object in the topos. Finally, the function symbols $A : \Sigma \to \mathcal{R}$ are represented by arrows betwen the objects Σ and $\mathcal{R}$ in the topos $\mathcal{E}$.

In classical physics, the representation is the obvious one: the topos to be used is the topos **Set** of sets and mappings, the symbol Σ is represented by a symplectic manifold $\mathcal{S}$ which is the state space, the symbol $\mathcal{R}$ is represented by the real numbers and function symbols $A : \Sigma \to \mathcal{R}$ are represented by real-valued functions $f_A : \mathcal{S} \to \mathbb{R}$. Propositions about physical quantities correspond to subsets of the state space.

3 The context category $\mathcal{V}(\mathcal{R})$ and the topos of presheaves $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$

We will now discuss the representation of $\mathcal{L}(S)$ in the case that S is to be described as a quantum system. We assume that S is a non-trivial system that −in the usual description− has a Hilbert space $\mathcal{H}$ of dimension 3 or greater, and that the physical quantities belonging to S form a von Neumann algebra $\mathcal{R}(S) \subseteq \mathcal{B}(\mathcal{H})$ that contains the identity operator $\widehat{1}$.[7]

From the Kochen-Specker theorem [KS67] we know that there is no state space model of quantum theory if the algebra of observables is $\mathcal{B}(\mathcal{H})$ (for the generalisation to von Neumann algebras see [Doe05]). More concretely, there is no state space $\mathcal{S}$ such that the physical quantities are real-valued functions on $\mathcal{S}$. The reason is that if there existed such

[7] There should arise no confusion between the von Neumann algebra $\mathcal{R} = \mathcal{R}(S)$ and the symbol $\mathcal{R}$ of our formal language, we hope.

a state space $\mathcal{S}$, then each point (i.e., state) $s \in \mathcal{S}$ would allow to assign values to all physical quantities at once, simply by evaluating the functions representing the physical quantities at s. One can show that under very mild and natural conditions, this leads to a mathematical contradiction.

For an *abelian* von Neumann algebra V, there is no such obstacle: the *Gel'fand spectrum* Σ_V of V can be interpreted as a state space, and the Gel'fand transforms $\overline{A}$ of self-adjoint operators $\widehat{A} \in V$, representing physical quantities, are real-valued functions on Σ_V. The Gel'fand spectrum Σ_V of an abelian von Neumann algebra V consists of the pure states λ on V (see e.g. [KR83]). Each $\lambda \in \Sigma_V$ also is a multiplicative state; for all $\widehat{A}, \widehat{B} \in V$, we have

$$\lambda(\widehat{A}\widehat{B}) = \lambda(\widehat{A})\lambda(\widehat{B}),$$

which, for projections $\widehat{P} \in \mathcal{P}(V)$, implies

$$\lambda(\widehat{P}) = \lambda(\widehat{P}^2) = \lambda(\widehat{P})\lambda(\widehat{P}) \in \{0, 1\}.$$

Finally, each $\lambda \in \Sigma_V$ is an algebra homomorphism from V to $\mathbb{C}$. The Gel'fand spectrum Σ_V is equipped with the weak* topology and thus becomes a compact Hausdorff space.

Let $\widehat{A} \in V$ and define

$$\begin{aligned} \overline{A} : \Sigma_V &\longrightarrow \mathbb{C} \\ \lambda &\longmapsto \overline{A}(\lambda) := \lambda(\widehat{A}). \end{aligned}$$

The function $\overline{A}$ is called the *Gel'fand transform* of $\widehat{A}$. It is a continuous function such that $\operatorname{im} \overline{A} = \operatorname{sp} \widehat{A}$. In particular, if $\widehat{A}$ is self-adjoint, then $\lambda(\widehat{A}) \in \operatorname{sp} \widehat{A} \subset \mathbb{R}$. The mapping

$$\begin{aligned} V &\longrightarrow C(\Sigma_V) \\ \widehat{A} &\longmapsto \overline{A} \end{aligned}$$

is called the *Gel'fand transformation* on V. It is an isometric $*$-isomorphism between V and $C(\Sigma_V)$.[8]

This leads to the idea of considering the set $\mathcal{V}(\mathcal{R})$ of non-trivial unital abelian von Neumann subalgebras of $\mathcal{R}$.[9] These abelian subalgebras are also called *contexts*. $\mathcal{V}(\mathcal{R})$ is partially ordered by inclusion and thus becomes a category. There is an arrow $i_{V'V} : V' \to V$ if and only if $V' \subseteq V$, and then $i_{V'V}$ is just the inclusion (or the identity arrow if $V' = V$). The category $\mathcal{V}(\mathcal{R})$ is called the *context category* and serves as our index category. The process of going from one abelian algebra V to a smaller algebra $V' \subset V$ can be seen as a process of *coarse-graining*: the algebra V' contains less physical quantities (self-adjoint operators), so we can describe less physics in V' than in V. We collect all the 'local state spaces' Σ_V into one large object:

[8]Of course, all this holds more generally for abelian C^*-algebras. We concentrate on von Neumann algebras, since we need these in our application.

[9]We exclude the trivial algebra $\mathbb{C}\widehat{1}$, which is a subalgebra of all other subalgebras.

Definition 2 *The* ***spectral presheaf*** $\underline{\Sigma}$ *is the presheaf*[10] *over* $\mathcal{V}(\mathcal{R})$ *defined*

a) on objects: for all $V \in \mathcal{V}(\mathcal{R})$, $\underline{\Sigma}_V = \Sigma_V$ *is the Gel'fand spectrum of* V,

b) on arrows: for all $i_{V'V}$, $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \to \underline{\Sigma}_{V'}$ *is given by restriction,* $\lambda \mapsto \lambda|_{V'}$.

The spectral presheaf was first considered by Chris Isham and Jeremy Butterfield in the series [IB98, IB99, IBH00, IB02] (see in particular the third of these papers). The presheaves over $\mathcal{V}(\mathcal{R})$ form a topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$. The arrows in this topos are natural transformations between the presheaves. Isham and Butterfield developed the idea that this is the appropriate topos for quantum theory. The object $\underline{\Sigma}$ in $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ serves as a state space analogue. In the light of the new developments in [DI07a]-[DI07d], using formal languages, we identify $\underline{\Sigma}$ as the state object in $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$, i.e., the representative of the symbol Σ of our formal language $\mathcal{L}(S)$.

Isham and Butterfield showed that the Kochen-Specker theorem is exactly equivalent to the fact that the spectral presheaf $\underline{\Sigma}$ has no elements, in the sense that there are no arrows from the terminal object $\underline{1}$ in $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ to $\underline{\Sigma}$. It is not hard to show that having an element of $\underline{\Sigma}$ would allow the assignment of real values to all physical quantities at once.

4 Representing $\mathcal{L}(S)$ in the presheaf topos $\mathrm{Set}^{\mathcal{V}(\mathcal{R})^{op}}$

The quantity-value object for quantum theory. We already have identified the topos for the quantum-theoretical description of a system S and the state object $\underline{\Sigma}$ in this topos. Let $V \in \mathcal{V}(\mathcal{R})$ be a context, then $\downarrow V := \{V' \in \mathcal{V}(\mathcal{R}) \mid V' \subseteq V\}$ denotes the set of all subalgebras of V, equipped with the partial order inherited from $\mathcal{V}(\mathcal{R})$. It can be shown that the symbol $\mathcal{R}$ should be represented by the following presheaf [DI07c]:

Definition 3 *The* ***presheaf*** $\underline{\mathbb{R}^{\leftrightarrow}}$ ***of order-preserving and -reversing functions*** *on* $\mathcal{V}(\mathcal{R})$ *is defined*

a) on objects: for all $V \in \mathcal{V}(\mathcal{R})$, $\underline{\mathbb{R}^{\leftrightarrow}}_V := \{(\mu, \nu) \mid \mu : \downarrow V \to \mathbb{R}$ *is order-preserving,* $\nu : \downarrow V \to \mathbb{R}$ *is order-reversing and* $\mu \leq \nu\}$,

b) on arrows: for all $i_{V'V}$, $\underline{\mathbb{R}^{\leftrightarrow}}(i_{V'V}) : \underline{\mathbb{R}^{\leftrightarrow}}_V \to \underline{\mathbb{R}^{\leftrightarrow}}_{V'}$ *is given by restriction,* $(\mu, \nu) \mapsto (\mu|_{V'}, \nu|_{V'})$.

Here, an order-preserving function $\mu : \downarrow V \to \mathbb{R}$ is a function such that $V'' \subseteq V'$ (where $V', V'' \in \downarrow V$) implies $\mu(V'') \leq \mu(V')$. Order-reversing functions are defined analogously.

The presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$ is *not* the real-number object $\underline{\mathbb{R}}$ in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$, which is the constant presheaf defined by $\underline{\mathbb{R}}(V) := \mathbb{R}$ for all V and $\underline{\mathbb{R}}(i_{V'V}) : \mathbb{R} \to \mathbb{R}$ as the identity. From the Kochen-Specker theorem, we would not expect that physical quantities take their values in the real numbers. (This does not mean that the results of measurements are not

[10] A presheaf is a contravariant functor from $\mathcal{V}(\mathcal{R})$ to $\mathbf{Set}$, and obviously, $\underline{\Sigma}$ is of this kind. In our notation, presheaves will always be underlined.

real numbers. We do not discuss measurement here.) More importantly, the presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$ takes into account the coarse-graining inherent in the base category $\mathcal{V}(\mathcal{R})$: at each stage V, a pair (μ, ν) consisting of an order-preserving and an order-reversing function defines a whole range or interval $[\mu(V), \nu(V)]$ of real numbers, not just a single real number. (It can happen that $\mu(V) = \nu(V)$.) If we go to a smaller subalgebra $V' \subset V$, which is a kind of coarse-graining, then we have $\mu(V') \leq \mu(V)$ and $\nu(V') \geq \nu(V)$, so the corresponding interval $[\mu(V'), \nu(V')]$ can only become larger.

The representation of function symbols $A : \Sigma \to \mathcal{R}$. In order to represent a physical quantity A belonging to the system S as an arrow from $\underline{\Sigma}$ to the presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$ of 'values', we have to use a two-step process.

1. Let $\widehat{A} \in \mathcal{R}$ be the self-adjoint operator representing A. For each abelian subalgebra $V \in \mathcal{V}(\mathcal{R})$, we define

$$\delta^o(\widehat{A})_V := \bigwedge\{\widehat{B} \in V_{sa} | \widehat{B} \geq_s \widehat{A}\},$$
$$\delta^i(\widehat{A})_V := \bigvee\{\widehat{C} \in V_{sa} | \widehat{C} \leq_s \widehat{A}\}.$$

Here, the *spectral order* on self-adjoint operators is used [Ols71, deG05]. This is defined for all self-adjoint operators $\widehat{A}, \widehat{B}$ with spectral families $\widehat{E}^A$ resp. $\widehat{E}^B$ as

$$\widehat{A} \leq_s \widehat{B} :\Leftrightarrow (\forall \lambda \in \mathbb{R} : \widehat{E}^A_\lambda \geq \widehat{E}^B_\lambda).$$

Equipped with the spectral order, the set of self-adjoint operators in a von Neumann algebra becomes a boundedly complete lattice. In particular, the mappings $\delta^o_V, \delta^i_V : \mathcal{R}_{sa} \to V_{sa}$ are well-defined. We call these mappings *outer* and *inner daseinisation*, respectively. The outer daseinisation $\delta^o(\widehat{A})_V$ of $\widehat{A}$ to the context V is the approximation from above by the smallest self-adjoint operator in V that is spectrally larger than $\widehat{A}$. Likewise, the inner daseinisation $\delta^i(\widehat{A})_V$ is the approximation from below by the largest self-adjoint operator in V that is spectrally smaller then $\widehat{A}$. Since the spectral order is coarser than the usual, linear order, we have, for all V,

$$\delta^i(\widehat{A})_V \leq \widehat{A} \leq \delta^o(\widehat{A})_V.$$

One can show that the spectra of $\delta^i(\widehat{A})_V$ and $\delta^o(\widehat{A})_V$ are subsets of the spectrum of $\widehat{A}$, which seems physically very sensible. If we used the approximation in the linear order, this would not hold in general. The approximation of self-adjoint operators in the spectral order was suggested by de Groote [deG05b, deG07]. If $V' \subset V$, then, by construction, $\delta^i(\widehat{A})_{V'} \leq_s \delta^i(\widehat{A})_V$ and $\delta^o(\widehat{A})_{V'} \geq_s \delta^o(\widehat{A})_V$, which implies

$$\delta^i(\widehat{A})_{V'} \leq \delta^i(\widehat{A})_V,$$
$$\delta^o(\widehat{A})_{V'} \geq \delta^o(\widehat{A})_V.$$

In this sense, the approximations to $\widehat{A}$ become coarser if the context becomes smaller.

2. Now that we have constructed a pair $(\delta^i(\widehat{A})_V, \delta^o(\widehat{A})_V)$ of operators approximating $\widehat{A}$ from below and from above for each context V, we can define a natural transformation $\breve{\delta}(\widehat{A})$ from $\underline{\Sigma}$ to $\underline{\mathbb{R}^{\leftrightarrow}}$ in the following way: let $V \in \mathcal{V}(\mathcal{R})$ be a context, and let $\lambda \in \underline{\Sigma}_V$ be a pure state of V. Then define, for all $V' \in \downarrow V$,

$$\mu_\lambda(V') := \lambda(\delta^i(\widehat{A})_{V'}) = \overline{\delta^i(\widehat{A})_{V'}}(\lambda),$$

where $\overline{\delta^i(\widehat{A})_{V'}}$ is the Gel'fand transform of the self-adjoint operator $\delta^i(\widehat{A})_{V'}$. From the theory of abelian C^*-algebras, it is known that $\lambda(\delta^i(\widehat{A})_{V'}) \in \mathrm{sp}(\delta^i(\widehat{A})_{V'})$ (see e.g. [KR83]). Let $V', V'' \in \downarrow V$ such that $V'' \subset V'$. We saw that $\delta^i(\widehat{A})_{V''} \leq \delta^i(\widehat{A})_{V'}$, which implies $\lambda(\delta^i(\widehat{A})_{V''}) \leq \lambda(\delta^i(\widehat{A})_{V'})$, so $\mu_\lambda : \downarrow V \rightarrow \mathbb{R}$ is an order-preserving function. Analogously, let

$$\nu_\lambda(V') := \lambda(\delta^o(\widehat{A})_{V'}) = \overline{\delta^o(\widehat{A})_{V'}}(\lambda)$$

for all $V' \in \downarrow V$. We obtain an order-reversing function $\nu_\lambda : \downarrow V \rightarrow \mathbb{R}$. Then, for all $V \in \mathcal{V}(\mathcal{R})$, let

$$\begin{aligned} \breve{\delta}(\widehat{A})(V) : \underline{\Sigma}_V &\longrightarrow \underline{\mathbb{R}^{\leftrightarrow}}_V \\ \lambda &\longmapsto (\mu_\lambda, \nu_\lambda). \end{aligned}$$

By construction, these mappings are the components of a natural transformation $\breve{\delta}(\widehat{A})$: $\underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$. For all $V, V' \in \mathcal{V}(\mathcal{R})$ such that $V' \subseteq V$, we have a commuting diagram

$$\begin{tikzcd}
\underline{\Sigma}_V \arrow[r, "\underline{\Sigma}(i_{V'V})"] \arrow[d, "\breve{\delta}(\widehat{A})(V)"'] & \underline{\Sigma}_{V'} \arrow[d, "\breve{\delta}(\widehat{A})(V')"] \\
\underline{\mathbb{R}^{\leftrightarrow}}_V \arrow[r, "\underline{\mathbb{R}^{\leftrightarrow}}(i_{V'V})"'] & \underline{\mathbb{R}^{\leftrightarrow}}_{V'}
\end{tikzcd}$$

The arrow $\breve{\delta}(\widehat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$ in the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ is the representative of the physical quantity A, which is abstractly described by the function symbol $A : \Sigma \rightarrow \mathcal{R}$ in our formal language. The physical content, namely the appropriate choice of the self-adjoint operator $\widehat{A}$ from which we construct the arrow $\breve{\delta}(\widehat{A})$, is not part of the language, but part of the representation.[11]

The representation of propositions. As discussed in subsection 1.2, in classical physics the subset of state space $\mathcal{S}$ representing a proposition "$A \in \Delta$" is constructed by taking the inverse image $f_A^{-1}(\Delta)$ of Δ under the function representing A. We will use the analogous construction in the topos formulation of quantum theory: the set Δ is a subset (that is, subobject) of the quantity-value object $\mathbb{R}$ in classical physics, so we start from

[11] The current scheme is not completely topos-internal yet. It is an open question if every arrow from $\underline{\Sigma}$ to $\underline{\mathbb{R}^{\leftrightarrow}}$ comes from a self-adjoint operator. This is why we start from a self-adjoint operator $\widehat{A}$ to construct $\breve{\delta}(\widehat{A})$. We are working on a more internal characterisation.

a subobject Θ of the presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$. We get a subobject of the state object $\underline{\Sigma}$ by pullback along $\breve{\delta}(\widehat{A})$, which we denote by $\breve{\delta}(\widehat{A})^{-1}(\Theta)$.[12] For details see subsection 3.6 in [DI07c] and also [HS07].

In both classical and quantum theory, propositions are represented by subobjects of the quantity-value object (state space $\mathcal{S}$ resp. spectral presheaf $\underline{\Sigma}$). Such subobjects are constructed by pullback from subobjects of the quantity-value object (real numbers $\mathbb{R}$ resp. presheaf of order-preserving and -reversing functions $\underline{\mathbb{R}^{\leftrightarrow}}$). The interpretation and meaning of such propositions is determined by the internal logic of the topos (**Set** resp. $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$). In the classical case, where **Set** is used, this is the ordinary Boolean logic that we are familiar with. In the quantum case, the internal logic of the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ has to be used. This intuitionistic logic can be interpreted using Kripke-Joyal semantics, see e.g. chapter VI in [MM92].

The Heyting algebra structure of subobjects. In the next section, we discuss the representation of states in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ and the assignment of truth-values to propositions. Before doing so, it is worth noting that the subobjects of $\underline{\Sigma}$ form a Heyting algebra (since the subobjects of any object in a topos do), so we have mapped propositions "$A \in \Delta$" (understood as discussed) to a *distributive* lattice with a pseudocomplement. Together with the results from the next section, we have a completely new form of quantum logic, based upon the internal logic of the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$. Since this is a distributive logic and since the internal logic of a topos has powerful rules of inference, this kind of quantum logic is potentially much better interpretable than ordinary quantum logic of the Birkhoff-von Neumann kind. The latter type of quantum logic and its generalisations are based on nondistributive structures and lack a deductive system.

5 Truth objects and truth-values

In classical physics, a state is just a point of state space.[13] Since, as we saw, the spectral presheaf $\underline{\Sigma}$ has no elements (or, global elements[14]), we must represent states differently in the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$.

5.1 Generalised elements as generalised states

One direct way, suggested in [HS07], is the following generalisation: $\underline{\Sigma}$ has no global elements $\underline{1} \to \underline{\Sigma}$, but it does have subobjects $\underline{U} \hookrightarrow \underline{\Sigma}$. In algebraic geometry and more generally in category theory, such monics (and, more generally, arbitrary arrows) are called

[12] This is a well-defined categorical construction, since the pullback of a monic is a monic, so we get a subobject of $\underline{\Sigma}$ from a subobject of $\underline{\mathbb{R}^{\leftrightarrow}}$.

[13] One might call this a *pure* state, though this is not customary in classical physics. Such a state actually is a point measure on state space, in contrast to more general probability measures that describe general states. We only consider pure states here and identify the point measure with the corresonding point of state space.

[14] Elements $\underline{1} \to \underline{P}$ of a presheaf $\underline{P}$ are called *global elements* or *global sections* in category theory. We follow this convention to avoid confusion with points or elements of sets.

generalised elements [LR03]. We could postulate that these subobjects, or some of them, are 'generalised states'. Consider another subobject of $\underline{\Sigma}$ that represents a proposition "$A \in \Delta$" about the quantum system, given by its characteristic arrow $\chi_{\underline{S}} : \underline{\Sigma} \to \underline{\Omega}$. Then we can compose these arrows

$$\underline{U} \hookrightarrow \underline{\Sigma} \to \underline{\Omega}$$

to obtain an arrow $\underline{U} \to \underline{\Omega}$. This is *not* a global element $\underline{1} \to \underline{\Omega}$ of $\underline{\Omega}$, and by construction, it cannot be, since $\underline{\Sigma}$ has no global elements, but it is a generalised element of $\underline{\Omega}$. It might be possible to give a physical meaning to these arrows $\underline{U} \to \underline{\Omega}$ if one can (a) give physical meaning to the subobject $\underline{U} \hookrightarrow \underline{\Sigma}$, making clear what a generalised state actually is, and (b) give a logical and physical interpretation of an arrow $\underline{U} \to \underline{\Omega}$. While a global element $\underline{1} \to \underline{\Omega}$ is interpreted as a truth-value in the internal logic of a topos, the logical interpretation of an arrow $\underline{U} \to \underline{\Omega}$ is not so clear.

We want to emphasise that mathematically, the above construction is perfectly well-defined. It remains to be worked out if a physical and logical meaning can be attached to it.

5.2 The construction of truth objects

We now turn to the construction of so-called 'truth objects' from pure quantum states ψ, see also [DI07b]. (To be precise, a unit vector ψ in the Hilbert space $\mathcal{H}$ represents a vector state $\varphi_\psi : \mathcal{R} \to \mathbb{C}$ on a von Neumann algebra, given by $\varphi_\psi(\widehat{A}) := \langle\psi|\,\widehat{A}\,|\psi\rangle$ for all $\widehat{A} \in \mathcal{R}$. If $\mathcal{R} = \mathcal{B}(\mathcal{H})$, then every φ_ψ is a pure state.) Of course, the Hilbert space $\mathcal{H}$ is the Hilbert space on which the von Neumann algebra of observables $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is represented. This is the most direct way in which Hilbert space actually enters the mathematical constructions inside the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$. However, we will see how this direct appeal to Hilbert space possibly can be circumvented.

Given a subobject of $\underline{\Sigma}$ that represents some proposition, a truth object will allow us to construct a global element $\underline{1} \to \underline{\Sigma}$ of $\underline{\Sigma}$, as we will show in subsection 5.4. This means that from a proposition and a state, we *do* get an actual truth-value for that proposition in the internal logic of the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$. The construction of truth objects is a direct generalisation of the classical case.

For the moment, let us consider sets. Let S be a subset of some larger set X, and let $x \in X$. Then

$$(x \in S) \Leftrightarrow (S \in U(x)),$$

where $U(x)$ denotes the set of neighbourhoods of x in X. The key observation is that while the l.h.s. cannot be generalised to the topos setting, since we cannot talk about points like x, the r.h.s. can. The task is to define neighbourhoods in a suitable manner. We observe that $U(x)$ is a subset of the power set $PX = P(X)$, which is the same as an element of the power set of the power set $PPX = P(P(X))$.

This leads to the idea that for each context $V \in \mathcal{V}(\mathcal{R})$, we must choose an appropriate set of subsets of the Gel'fand spectrum $\underline{\Sigma}_V$ such that these sets of subsets form an element in $PP\underline{\Sigma}$. Additionally, the subsets we choose at each stage V should be clopen, since the clopen subsets $P_{cl}(\underline{\Sigma}_V)$ form a lattice that is isomorphic to the lattice $\mathcal{P}(V)$ of projections in V.

The main difficulty lies in the fact that the spectral presheaf $\underline{\Sigma}$ has no global elements, which is equivalent to the Kochen-Specker theorem. A global element, if it existed, would pick one point λ_V from each Gel'fand spectrum $\underline{\Sigma}_V$ ($V \in \mathcal{V}(\mathcal{R})$) such that, whenever $V' \subset V$, we would have $\lambda_{V'} = \lambda_V|_{V'}$. If we had such global elements, we could define neighbourhoods for them by taking, for each $V \in \mathcal{V}(\mathcal{R})$, neighbourhoods of λ_V in $\underline{\Sigma}_V$.

Since no such global elements exist, we cannot expect to have neighbourhoods of *points* at each stage. Rather, we will get neighbourhoods of *sets* at each stage V, and only for particular V, these sets will have just one element. In any case, the sets will depend on the state ψ in a straighforward manner. We define:

Definition 4 *Let $\psi \in \mathcal{H}$ be a unit vector, let $\widehat{P}_\psi$ the projection onto the corresponding one-dimensional subspace (i.e., ray) of $\mathcal{H}$, and let $P_{cl}(\underline{\Sigma}_V)$ be the clopen subsets of the Gel'fand spectrum $\underline{\Sigma}_V$. If $S \in P_{cl}(\underline{\Sigma}_V)$, then $\widehat{P}_S \in \mathcal{P}(V)$ denotes the corresponding projection. The truth object $\mathbb{T}^\psi = (\mathbb{T}^\psi_V)_{V \in \mathcal{V}(\mathcal{R})}$ is given by*

$$\forall V \in \mathcal{V}(\mathcal{R}) : \mathbb{T}^\psi_V := \{S \in P_{cl}(\underline{\Sigma}_V) \mid \langle\psi|\, \widehat{P}_S \,|\psi\rangle = 1\}.$$

At each stage V, $\mathbb{T}^\psi_V$ collects all subsets S of $\underline{\Sigma}_V$ such that the expectation value of the projection corresponding to this subset is 1. From this definition, it is not clear at first sight that the set $\mathbb{T}^\psi_V$ can be seen as a set of neighbourhoods.

Lemma 5 *We have the following equalities:*

$$\begin{aligned}\forall V \in \mathcal{V}(\mathcal{R}) : \mathbb{T}^\psi_V &= \{S \in P_{cl}(\underline{\Sigma}_V) \mid \langle\psi|\, \widehat{P}_S \,|\psi\rangle = 1\}\\ &= \{S \in P_{cl}(\underline{\Sigma}_V) \mid \widehat{P}_S \geq \widehat{P}_\psi\}\\ &= \{S \in P_{cl}(\underline{\Sigma}_V) \mid \widehat{P}_S \geq \delta^o(\widehat{P}_\psi)_V\}\\ &= \{S \in P_{cl}(\underline{\Sigma}_V) \mid S \supseteq S_{\delta^o(\widehat{P}_\psi)_V}\}.\end{aligned}$$

Proof. If $\langle\psi|\, \widehat{P}_S \,|\psi\rangle = 1$, then ψ lies entirely in the subspace of Hilbert space that $\widehat{P}_S$ projects onto. This is equivalent to $\widehat{P}_S \geq \widehat{P}_\psi$. Since $\widehat{P}_S \in \mathcal{P}(V)$ and $\delta^o(\widehat{P}_\psi)_V$ is the *smallest* projection in V that is larger than $\widehat{P}_\psi$,[15] we also have $\widehat{P}_S \geq \delta^o(\widehat{P}_\psi)_V$. In the last step, we simply go from the projections in V to the corresponding clopen subsets of $\underline{\Sigma}_V$. ■

This reformulation shows that $\mathbb{T}^\psi_V$ actually consists of subsets of the Gel'fand spectrum $\underline{\Sigma}_V$ that can be seen as some kind of neigbourhoods, not of a single point of $\underline{\Sigma}_V$, but of a certain subset of $\underline{\Sigma}_V$, namely $S_{\delta^o(\widehat{P}_\psi)_V}$. In the simplest case, we have $\widehat{P}_\psi \in \mathcal{P}(V)$, so

[15]On projections, the spectral order $\leq_s$ and the linear order $\leq$ coincide.

$\delta^o(\widehat{P}_\psi)_V = \widehat{P}_\psi$. Then $S_{\delta^o(\widehat{P}_\psi)_V} = S_{\widehat{P}_\psi}$, and this subset contains a single element, namely the pure state λ such that

$$\lambda(\widehat{P}_\psi) = 1$$

and $\lambda(\widehat{Q}) = 0$ for all $\widehat{Q} \in \mathcal{P}(V)$ such that $\widehat{Q}\widehat{P}_\psi = 0$. In this case, $\mathbb{T}^\psi_V$ actually consists of all the clopen neighbourhoods of the point λ in $\underline{\Sigma}_V$.

In general, if $\widehat{P}_\psi$ does not lie in the projections $\mathcal{P}(V)$, then there is no subset of $\underline{\Sigma}_V$ that corresponds directly to $\widehat{P}_\psi$. We must first approximate $\widehat{P}_\psi$ by a projection in V, and $\delta^o(\widehat{P}_\psi)_V$ is the smallest projection in V larger than $\widehat{P}_\psi$. The projection $\delta^o(\widehat{P}_\psi)_V$ corresponds to a subset $S_{\delta^o(\widehat{P}_\psi)_V} \subseteq \underline{\Sigma}_V$ that may contain more than one element. However, $\mathbb{T}^\psi_V$ can still be seen as a set of neighbourhoods, but now of this set $S_{\delta^o(\widehat{P}_\psi)_V}$ rather than of a single point.

It is an interesting and non-trivial point that the (outer) daseinisation $\delta^o(\widehat{P}_\psi)_V$ ($V \in \mathcal{V}(\mathcal{R})$) shows up in this construction. We did not discuss this here, but the subobjects of $\underline{\Sigma}$ constructed from the outer daseinisation of projections play a central rôle in the representation of a certain propositional language $\mathcal{PL}(S)$ that one can attach to a physical system S [DI07a, DI07b]. Moreover, these subobjects are 'optimal' in the sense that, whenever $V' \subset V$, the restriction from $S_{\delta^o(\widehat{P})_V}$ to $S_{\delta^o(\widehat{P})_{V'}}$ is *surjective*, see Theorem 3.1 in [DI07b]. This property can also lead the way to a more internal characterisation of truth-objects, without reference to a state ψ and hence to Hilbert space.

5.3 Truth objects and Birkhoff-von Neumann quantum logic

There is yet another point of view on what a truth object $\mathbb{T}^\psi$ is, closer the ordinary quantum logic, which goes back to the famous paper [BV36] by Birkhoff and von Neumann. For now, let us assume that $\mathcal{R} = \mathcal{B}(\mathcal{H})$, then we write $\mathcal{P}(\mathcal{H}) := \mathcal{P}(\mathcal{B}(\mathcal{H}))$ for the lattice of projections on Hilbert space. In their paper, Birkhoff and von Neumann identify a proposition "$A \in \Delta$" about a quantum system with a projection operator $\widehat{E}[A \in \Delta] \in \mathcal{P}(\mathcal{H})$ via the spectral theorem [KR83] and interpret the lattice structure of $\mathcal{P}(\mathcal{H})$ as giving a quantum logic. This is very different from the topos form of quantum logic, since $\mathcal{P}(\mathcal{H})$ is a *non-distributive* lattice, leading to all the well-known interpretational difficulties. Nonetheless, in this subsection we want to interpret truth objects from the perspective of Birkhoff-von Neumann quantum logic.

The implication in ordinary quantum logic is given by the partial order on $\mathcal{P}(\mathcal{H})$: a proposition "$A \in \Delta_1$" implies a proposition "$B \in \Delta_2$" (where we can have $B = A$) if and only if $\widehat{E}[A \in \Delta_1] \leq \widehat{E}[B \in \Delta_2]$ holds for the corresponding projections.

The idea now is that, given a pure state ψ and the corresponding projection $\widehat{P}_\psi$ onto a ray, we can collect all the projections larger than or equal to $\widehat{P}_\psi$. We denote this by

$$T^\psi := \{\widehat{P} \in \mathcal{P}(\mathcal{H}) \mid \widehat{P} \geq \widehat{P}_\psi\}.$$

The propositions represented by these projections are exactly those propositions about the quantum system that are (totally) true if the system is in the state ψ. Totally true means 'true with probability 1' in an instrumentalist interpretation. If, for example, a projection $\widehat{E}[A \in \Delta]$ is larger than $\widehat{P}_\psi$ and hence contained in T^ψ, then, upon measurement of the physical quantity A, we will find the measurement result to lie in the set Δ with certainty (i.e., with probability 1).

T^ψ is a maximal (proper) filter in $\mathcal{P}(\mathcal{H})$. Every pure state ψ gives rise to such a maximal filter T^ψ, and clearly, the mapping $\psi \mapsto T^\psi$ is injective. We can obtain the truth object $\mathbb{T}^\psi$ from the maximal filter T^ψ simply by defining

$$\forall V \in \mathcal{V}(\mathcal{R}) : \mathbb{T}^\psi_V := T^\psi \cap V.$$

In each context V, we collect all the projections larger than $\widehat{P}_\psi$. On the level of propositions, we have all the propositions about physical quantities A *in the context* V that are totally true in the state ψ.

5.4 The assignment of truth-values to propositions

We return to the consideration of the internal logic of the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ and show how to define a global element $\underline{1} \to \underline{\Omega}$ of the subobject classifier from a clopen subobject $\underline{S}$ of $\underline{\Sigma}$ and a truth object $\mathbb{T}^\psi$. The subobject $\underline{S}$ represents a proposition about the quantum system, the truth object $\mathbb{T}^\psi$ represents a state, and the global element of $\underline{\Omega}$ will be interpreted as the truth-value of the proposition in the given state. Thus, we make use of the internal logic of the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ of presheaves over the context category $\mathcal{V}(\mathcal{R})$ to assign truth-values to all propositions about a quantum system.

It is well known that the subobject classifier $\underline{\Omega}$ in a topos of presheaves is the presheaf of *sieves* (see e.g. [MM92]). A sieve σ on an object A in some category $\mathcal{C}$ is a collection of arrows with codomain A with the following property: if $f : B \to A$ is in σ and $g : C \to B$ is another arrow in $\mathcal{C}$, then $f \circ g : C \to A$ is in σ, too. In other words, a sieve on A is a downward closed set of arrows with codomain A. Since the context category $\mathcal{V}(\mathcal{R})$ is a partially ordered set, things become very simple: the only arrows with codomain V are the inclusions $i_{V'V}$. Since such an arrow is specified uniquely by its domain V', we can think of the sieve σ on V as consisting of certain subalgebras V' of V. If $V' \in \sigma$ and $V'' \subset V'$, then $V'' \in \sigma$.

The restriction mappings of the presheaf $\underline{\Omega}$ are given by pullbacks of sieves. The pullback of sieves over a partially ordered set takes a particularly simple form:

Lemma 6 *If σ is a sieve on $V \in \mathcal{V}(\mathcal{R})$ and $V' \subset V$, then the pullback $\sigma \cdot i_{V'V}$ is given by $\sigma \cap \downarrow V'$. (This holds analogously for sieves on any partially ordered set, not just $\mathcal{V}(\mathcal{R})$).*

Proof. For the moment, we switch to the arrows notation. By definition, the pullback $\sigma \cdot i_{V'V}$ is given by

$$\sigma \cdot i_{V'V} := \{ i_{V''V'} \mid i_{V'V} \circ i_{V''V'} \in \sigma \}.$$

We now identify arrows and subalgebras as usual and obtain (using the fact that $V'' \subseteq V'$ implies $V'' \subset V$)

$$\{i_{V''V'} \mid i_{V'V} \circ i_{V''V'} \in \sigma\} \simeq \{V'' \subseteq V' \mid V'' \in \sigma\} = \downarrow\! V' \cap \sigma.$$

Since $\downarrow V'$ is the maximal sieve on V', the pullback $\sigma \cdot i_{V'V}$ is given as the intersection of σ with the maximal sieve on V'. ■

The *name* $\ulcorner \underline{S} \urcorner$ of the subobject $\underline{S}$ is the unique arrow $\underline{1} \to P\underline{\Sigma} = \underline{\Omega}^{\underline{\Sigma}}$ into the power object of $\underline{\Sigma}$ (i.e., the subobjects of $\underline{\Sigma}$) that 'picks out' $\underline{S}$ among all subobjects. $\ulcorner \underline{S} \urcorner$ is a global element of $P\underline{\Sigma}$. Here, one uses the fact that power objects behave like sets, in particular, they have global elements. Since we assume that $\underline{S}$ is a *clopen* subobject, we also get an arrow $\underline{1} \to P_{cl}\underline{\Sigma}$ into the clopen power object of $\underline{\Sigma}$, see [DI07b]. We denote this arrow by $\ulcorner \underline{S} \urcorner$, as well.

Since $\mathbb{T}^{\psi} \in PP_{cl}\underline{\Sigma}$ is a collection of clopen subobjects of $\underline{\Sigma}$, it makes sense to ask if $\underline{S}$ is among them; an expression like $\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi}$ is well-defined. We define, for all $V \in \mathcal{V}(\mathcal{R})$, the *valuation*

$$v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V := \{V' \subseteq V \mid \underline{S}(V') \in \mathbb{T}^{\psi}_{V'}\}.$$

At each stage V, we collect all those subalgebras of V such that $\underline{S}(V')$ is contained in $\mathbb{T}^{\psi}_{V'}$.

In order to construct a global element of the presheaf of sieves $\underline{\Omega}$, we must first show that $v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$ is a sieve on V. In the proof we use the fact that the subobjects obtained from daseinisation are optimal in a certain sense.

Proposition 7 *$v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V := \{V' \subseteq V \mid \underline{S}(V') \in \mathbb{T}^{\psi}_{V'}\}$ is a sieve on V.*

Proof. As usual, we identify an inclusion morphism $i_{V'V}$ with V' itself, so a sieve on V consists of certain subalgebras of V. We have to show that if $V' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$ and $V'' \subset V'$, then $V'' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$. Now, $V' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$ means that $\underline{S}(V') \in \mathbb{T}^{\psi}_{V'}$, which is eqivalent to $\underline{S}(V') \supseteq \underline{S}_{\delta^o(\widehat{P}_{\psi})_{V'}}$. Here, $\underline{S}_{\delta^o(\widehat{P}_{\psi})_{V'}}$ is the component at V' of the sub-object $\underline{S}_{\delta^o(\widehat{P}_{\psi})} = (\underline{S}_{\delta^o(\widehat{P}_{\psi})_V})_{V \in \mathcal{V}(\mathcal{R})}$ of $\underline{\Sigma}$ obtained from daseinisation of $\widehat{P}_{\psi}$. According to Thm. 3.1 in [DI07b], the sub-object $\underline{S}_{\delta^o(\widehat{P}_{\psi})}$ is optimal in the following sense: when restricting from V' to V'', we have $\underline{\Sigma}(i_{V''V'})(\underline{S}_{\delta^o(\widehat{P}_{\psi})_{V'}}) = \underline{S}_{\delta^o(\widehat{P}_{\psi})_{V''}}$, i.e., the restriction is surjective. By assumption, $\underline{S}(V') \supseteq \underline{S}_{\delta^o(\widehat{P}_{\psi})_{V'}}$, which implies

$$\underline{S}(V'') \supseteq \underline{\Sigma}(i_{V''V'})(\underline{S}(V')) \supseteq \underline{\Sigma}(i_{V''V'})(\underline{S}_{\delta^o(\widehat{P}_{\psi})_{V'}}) = \underline{S}_{\delta^o(\widehat{P}_{\psi})_{V''}}.$$

This shows that $\underline{S}(V'') \in \mathbb{T}^{\psi}_{V''}$ and hence $V'' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$. ■

Finally, we have to show that the sieves $v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$, $V \in \mathcal{V}(\mathcal{R})$, actually form a global element of $\underline{\Omega}$, i.e., they all fit together under the restriction mappings of the presheaf $\underline{\Omega}$:

Proposition 8 *The sieves $v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^{\psi})_V$, $V \in \mathcal{V}(\mathcal{R})$, (see Prop. 7) form a global element of $\underline{\Omega}$.*

Proof. From Lemma 6, is suffices to show that, whenever $V' \subset V$, we have $v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_{V'} = v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_V \cap \downarrow V'$. If $V'' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_{V'}$, then $\underline{S}(V'') \in \mathbb{T}^\psi_{V''}$, which implies $V'' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_V$. Conversely, if $V'' \in \downarrow V'$ and $V'' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_V$, then, again, $\underline{S}(V'') \in \mathbb{T}^\psi_{V''}$, which implies $V'' \in v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_{V'}$. ■

The global element $v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi) = (v(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_V)_{V \in \mathcal{V}(\mathcal{R})}$ of $\underline{\Omega}$ is interpreted as the truth-value of the proposition represented by $\underline{S} \in P_{cl}(\underline{\Sigma})$ if the quantum system is in the state ψ (resp. $\mathbb{T}^\psi$). This assignment of truth-values is

- contextual, since the contexts $V \in \mathcal{V}(\mathcal{R})$ play a central rôle in the whole construction
- global in the sense that *every* proposition is assigned a truth-value
- completely independent of any notion of measurement or observer, hence we call our scheme a 'neo-realist' formulation of quantum theory
- topos-internal, the logical structure is not chosen arbitrarily, but fixed by the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$. This topos is directly motivated from the Kochen-Specker theorem
- non-Boolean, since there are (a) more truth-values than just *'true'* and *'false'* and (b) the global elements form a *Heyting* algebra, not a Boolean algebra. There is a global element 1 of $\underline{\Omega}$, consisting of the maximal sieve $\downarrow V$ at each stage V, which is interpreted as 'totally true', and there is a global element 0 consisting of the empty sieve for all V, which is interpreted as 'totally false'. Apart from that, there are many other global elements that represent truth-values between 'totally true' and 'totally false'. These truth-values are neither numbers nor probabilities, but are given by the logical structure of the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$. Since a Heyting algebra in particular is a partially ordered set, there are truth-values v_1, v_2 such that neither $v_1 < v_2$ nor $v_2 < v_1$, which is also different from two-valued Boolean logic where simply $0 < 1$ (i.e., 'false'<'true'). The presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ has a rich logical structure.

6 Conclusion and outlook

The formulation of quantum theory within the presheaf topos $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ gives a theory that is remarkably similar to classical physics from a structural perspective. In particular, there is a state object (the spectral presheaf $\underline{\Sigma}$) and a quantity-value object (the presheaf $\underline{\mathbb{R}^{\leftrightarrow}}$ of order-preserving and -reversing functions). Physical quantities are represented by arrows between $\underline{\Sigma}$ and $\underline{\mathbb{R}^{\leftrightarrow}}$.

One of the future tasks will be the incorporation of dynamics. The process of daseinisation behaves well with respect to the action of unitary operators, see section 5.2 in [DI07c], so it is conceivable that there is a 'Heisenberg picture' of dynamics. Commutators remain to be understood in the topos picture. On the other hand, it is possible to let a truth-object $\mathbb{T}^\psi$ change in time by applying Schrödinger evolution to ψ. It remains to be shown how this can be understood topos-internally.

Mulvey and Banaschewski have recently shown how to define the Gel'fand spectrum of an abelian C^*-algebra $\mathfrak{A}$ in any Grothendieck topos, using constructive methods (see [BM06] and references therein). Spitters and Heunen made the following construction in [HS07]: one takes a *non-abelian* C^*-algebra $\mathfrak{A}$ and considers the topos of (covariant) functors over the category of abelian subalgebras of $\mathfrak{A}$. The algebra $\mathfrak{A}$ induces an internal *abelian* C^*-algebra $\underline{\mathfrak{A}}$ in this topos of functors. (Internally, algebraic operations are only allowed between commuting operators.) Spitters and Heunen observed that the Gel'fand spectrum of this internal algebra basically is the spectral presheaf.[16] It is very reassuring that the spectral presheaf not only has a physical interpretation, but also such a nice and natural mathematical one. Spitters and Heunen also discuss integration theory in the constructive context. These tools will be very useful in order to regain actual numbers and expectation values from the topos formalism.

Since the whole topos programme is based on the representation of formal languages, major generalisations are possible. One can represent the same language $\mathcal{L}(S)$ in different topoi, as we already did with **Set** for classical physics and $\mathbf{Set}^{\mathcal{V}(\mathcal{R})^{op}}$ for algebraic quantum theory. For physical theories going beyond this, other topoi will play a rôle. The biggest task is the incorporation of space-time concepts, which will, at the very least, necessitate a change of the base category. It is also conceivable that the 'smooth topoi' of synthetic differential geometry (SDG) will play a rôle.

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